Magnetic Analysis by a Flux-Centered Approach
Part 1: Choosing Flux Linkage As An Independent Variable

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Executive Summary

Ferromagnetic circuit analysis provides an engineering tool for the design and control of electric motors and solenoids. In an approximate but useful form of the analysis, magnetic hysteresis is ignored, and eddy currents are approximated by a "shorted-turn" circuit model. The analysis of this paper relies on these approximations but allows for the later introduction of magnetic saturation, whose effects can be critical in solenoid analysis.

For analyzing the dynamic performance of a solenoid or motor, the characteristics of the device are measured in advance and subsequently treated as constraining relationships. Thus, inductance may be treated as a function of measured armature position, or position may be treated as a function of measured inductance. When a particular variable is chosen as an independent variable (e.g., position), then other variables (e.g., inductance, reluctance) are constrained as dependent variables.

Ignoring hysteresis and using a shorted-turn eddy current model, the dynamic state of a solenoid or motor is totally specified by three independent variables, called the <u>state variables</u>. A common choice is 1) winding current, 2) armature velocity (linear or angular), and 3) armature position (linear or angular). Given these choices, the list of dependent variables includes: inductance, reluctance, flux linkage, magnetic force, and velocity. A third category of parameters describes external interactions: applied voltage and external force.

Several alternative sets of three state variables can be used with equal validity to specify the system. Certain choices lead to simpler formulations, with greater potential for insight and further innovation – the "equally valid" solar system models of Ptolemy (earth at the center) and Copernicus (sun at the center) come to mind. In the case of solenoids transitioning from low inductance to much higher inductance as they close, it is argued that the state vector of (flux-linkage, velocity, position) is a preferable representation to the state vector of (current, velocity, position). The dynamic equation for flux-linkage approaches a linear differential equation as a solenoid approaches magnetic closure, while the comparable dynamic equation for current involves two competing terms with singularities just beyond the landing position: inductance, and back-EMF. Unlike the troublesome dynamic equation, the equation for current as a non-dynamic dependent variable is free of singularities.

"Sensorless" servo control is based on voltage and current measurement and mathematical inference of flux linkage and position. It is argued that this approach can work better than position measurement for control of solenoid soft-landing. In the "sensorless" approach, using control of flux linkage rather than current, the magnetic force is controlled with lower error sensitivity near landing, and magnetic saturation effects are always known and under control, with no inadvertent current spiking.

Building on this paper, a Part 2 follow-up will provide more complete derivations and will introduce a well-behaved algebraic expression for reciprocal-inductance as a function of position, describing a large class of solenoids with surprising accuracy. Future papers will introduce saturation models and approximate corrections for hysteresis and eddy currents. These simple models complement the powerful but cumbersome tools of FEA, providing a bridge from FEA to equation models useful in dynamic simulation and control.

The "Classic" Formulation

This paper deals with electromagnetic circuit analysis and simulation as applied to solenoids and motors. For simulation, the dynamic state variables are usually chosen to be "momentum like" quantities, that is, cumulative variables resulting from the integration of some force-like variable. Electric current, armature velocity, and armature position are the "classic" state variables for solenoid analysis. A similar formulation applies to rotary motors, substituting "angle" for "position." The dynamic system is driven by voltage on the electrical side and by force on the mechanical side. The electrical side has an inertia-like quantity called inductance, while the mechanical side has an inertia-like quantity called mass. For losses, one has electrical resistance and friction damping. Finally looking inside the "black box," one recognizes some variable internal parameters: magnetic reluctance, magnetic flux linkage, and mechanical momentum. If the nature of the solenoid is specified in advance, then these internal parameters may be expressed as dependent variables.

Thus, for a given solenoid, magnetic reluctance can be described as a function of armature position. Given that descriptive function and a specified position, X, then reluctance $\Re$ and inductance L become dependent variables:

$\Re = \Re(X)$ … symbolic statement that "Reluctance $\Re$ is a function of position X."

$L = L(X)$ … symbolic statement that "Inductance L is a function of position X."

Thus, both $\Re$ and L fall into the dependent variable category if X is chosen as the independent variable.

Familiar Dynamic State Variables:
  1]   I = winding current   … from integral of voltage, also dependent on position
  2] $\dot{X}$ = armature velocity   … from integral of force/mass
  3]   X = armature position   … from integral of velocity

If the characteristics of the solenoid are quantified in advance, this characterization leads to several dependent variables, some of which might swap places with state variables.

Familiar Dependent Variables:
  4] $\Re$ = magnetic reluctance
  5]   L = inductance
  6] $n\Phi$ = flux linkage (a common notation: $\lambda = n\Phi$)
  7]   F = magnetic force

Fixed Characteristics of the Solenoid:
  8]   R = electrical resistance (may vary significantly with temperature)
  9]   n = number of windings
 10]   M = moving mass
 11]   K = spring rate
 12] X_{K0} = spring equilibrium
 13]   D = damping coefficient

Solenoid Characteristic Equations:

14] $L = L(X)$ inductance is a calibratable function of position

15] $\mathfrak{R} = \mathfrak{R}(X)$ magnetic reluctance is a calibratable function of position

External Influences:

16] V = applied voltage

17] F = load force

Familiar Equations:

18] $I L = n\Phi \ (= \lambda)$ "inductive momentum" is analogous to mechanical momentum

19a] $E_i = \frac{1}{2} I^2 L$

 or   "inductive energy" is in balance with mechanical energy

19b] $E_i = \frac{1}{2} I n\Phi$

20] $V_i = V - I R$ "inductive voltage" after correction for resistive voltage drop

21a] $V_i = \dfrac{d(I L)}{dt}$

 or   inductive voltage is the "force" that changes inductive momentum

21b] $V_i \equiv \dfrac{d(n\Phi)}{dt}$

22] $n\Phi \equiv \int V_i \, dt + \text{const.}$ Eq. 21b or 22 <u>defines</u> flux linkage

　　　Where does Eq. 18 come from? How do we know that the flux linkage $n\Phi$ is identical to the product of current and inductance? First, note that the expression "$n\Phi$" is treated as a single variable, not as the product of two variables, "n" and "Φ". In fact, "$n\Phi$", called "λ" in some papers, is defined equivalently by Eq. 21b or Eq. 22. To derive some of the relationships in Eqs. 18 through 22, we repeat a few fundamental formulas. Since the context of these formulas is somewhat different, we use the numbering sequence "f1, f2, etc." to set these formulas apart from the later sequence continuing with Eq. 23.

　　　A given magnetic flux quantity, Φ, is meaningful only when associated with a specific cross-sectional area. For a one-turn winding, Maxwell's equations imply that:

f1] $V_i = \dfrac{d\Phi}{dt}$ fundamental formula #1, when flux Φ links <u>one turn</u>

　　　For a winding with n turns, the flux is not the same for all the turns. The net flux linkage variable defined in Eqs. 21b and 22 is named "$n\Phi$" to suggest the cumulative flux effect on n windings. For fixed inductors, inductance is <u>defined</u> as the coefficient that relates the time derivative of current to inductive voltage:

f2] $V_i |_L = L \dfrac{dI}{dt}$ fundamental formula #2 = definition of <u>fixed</u> inductance L

The physics books show that by multiplying both sides of Eq. f2 by current "I" one gets power, which is the time derivative of inductive magnetic energy E_i:

f3] $\dfrac{\partial E_i}{\partial t}\Big|_L = V_i\big|_L \bullet I$ time derivative of inductive energy at constant L

f4] $\dfrac{\partial E_i}{\partial t}\Big|_L = L \bullet \dfrac{dI}{dt} \bullet I$ substituting from f2 in f3

f5] $\dfrac{\partial E_i}{\partial t}\Big|_L = L \bullet \dfrac{d(\ I^2)}{dt}$ from f4

Integrating both sides of f5 with respect to time, starting from zero energy at zero current, one obtains the textbook formula for energy in a fixed inductor:

f6] $E_i = \tfrac{1}{2}\,I^2\,L$ "textbook" inductive energy formula, confirms Eq. 19a

Returning to the beginning of the energy derivation at f3, one could substitute from Eq. 21b instead of from the f2 definition:

f3] $\dfrac{\partial E_i}{\partial t}\Big|_L = V_i\big|_L \bullet I$ f3 again, now starting a flux linkage derivation

f7] $\dfrac{\partial E_i}{\partial t}\Big|_L = \dfrac{d(n\Phi)}{dt} \bullet I$ substituting this time from Eq. 21b

f8] $\dfrac{\partial E_i}{\partial t}\Big|_L = \dfrac{d(n\Phi)}{dt} \bullet n\Phi \bullet \dfrac{I}{n\Phi}$ multiplying and dividing by $n\Phi$

f9] $\dfrac{\partial E_i}{\partial t}\Big|_L = \dfrac{d(\ (n\Phi)^2)}{dt} \bullet \dfrac{I}{n\Phi}$ from f8

A fundamental physics equation is needed here:

f10] $\dfrac{I}{n\Phi} = \text{const.}$ special case for constant X and constant L

The concept conveyed by f10 is that in a linear inductor with constant geometry, current and flux will change everywhere in linear proportion. Consequently, the particular flux pattern linking the winding and causing the flux linkage is linear with current. Integrating both sides of f9 with respect to time with " $\dfrac{I}{n\Phi}$ " behaving like a constant coefficient, yields:

f11] $E_i = \tfrac{1}{2}(n\Phi)^2 \bullet \dfrac{I}{n\Phi}$ from f9 and f10

f12] $E_i = \tfrac{1}{2}I\,n\Phi$ simplifying f11, this confirms Eq. 19b

Returning to the earlier question, where did Eq. 18 come from? The derivations of Eqs. f6 and f12 confirm Eqs. 19a and 19b. Setting the right sides of Eqs. 19a and 19b equal to each other and simplifying yields Eq. 18. Eq. 21b was a definition of flux linkage, being a natural extension of the fundamental formula Eq. f1 when one considers a winding of n turns rather than flux through a one-turn loop of wire. Eq. 21a follows from Eqs. 18 and 21b.

The Flux Linkage Formulation

Flux linkage "$n\Phi$" was introduced in Eq. 6 as a dependent variable. Eq. 18 shows that this dependent variable is the product of the state variable "I" multiplied by the dependent variable "L". Alternatively, one could make "$n\Phi$" a state variable and then solve for the dependent variable "I", giving us new choices for state variables and independent variables.

Flux-based Dynamic State Variables:
  1'] $n\Phi$ = flux linkage   … from integral of inductive voltage V_i (see Eq. 20)
  2'] $\dot{X}$ = armature velocity   … from integral of force/mass
  3']   X = armature position   … from integral of velocity

Flux-based Dependent Variables:
  4'] $\mathfrak{R}$ = magnetic reluctance   = $\mathfrak{R}(X)$ $\mathfrak{R}$ is a calibrated function of X
  5']   1/L = reciprocal inductance   = $\mathfrak{R}/n^2$ formula for "n" windings
  6']   I = winding current   = $n\Phi\,(1/L)$
  7']   F = magnetic force   = $-\frac{1}{2}(n\Phi)^2\,\dfrac{\partial(1/L)}{\partial X}$

Comparing to the original equations 1 through 7, Eqs. 1' and 6' are in the swapped positions of the original Eqs. 1 and 6. Furthermore, the inductance "L" of Eq. 5 becomes reciprocal inductance "1/L" in Eq. 5'. As indicated, 1/L varies in linear proportion to magnetic reluctance "$\mathfrak{R}$". Suppose the position variable X is defined as the magnetic gap in a solenoid, so that X = 0 corresponds to the position of maximum mechanical closure, maximum inductance, and minimum magnetic reluctance. The appearance of the function 1/L (X) is shown in Fig. 1, graphed from measurements on a small engine valve actuation solenoid using E-core laminations attracting a slab of armature laminations:

Fig. 1
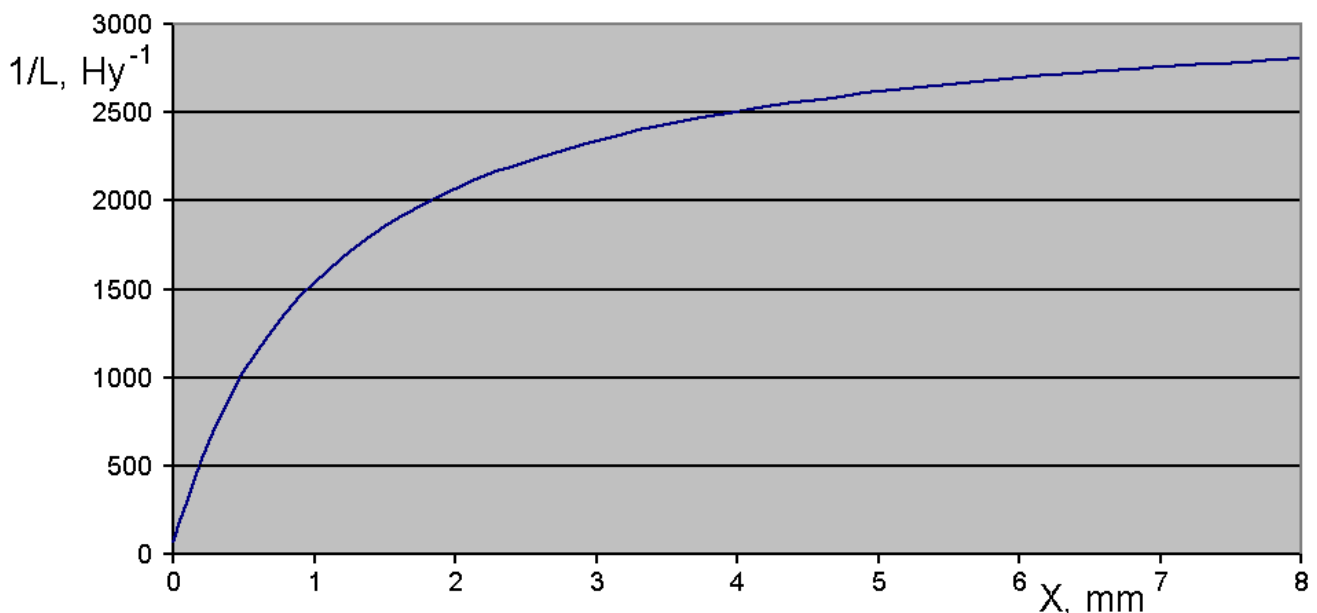
The reluctance function, $\mathfrak{R}(X)$, has the same appearance with a different vertical scale factor. The $1/L$ (X) function goes almost to zero at $X = 0$. While L is rising steeply toward a singularity in the inaccessible region just to the left of the origin, the function $1/L$ is gently curving with no sign of a singularity. For a solenoid whose armature and yoke reach full contact with no remaining magnetic gaps, the tangent slope to the $1/L$ function at $X = 0$ can be estimated quite accurately by a formula involving the mating magnetic poleface area, the slope (if any) of that area relative to the direction of axial motion (for example, the slope of a conical armature end surface compared to the zero-slope of a flat-ended armature), and the number of windings, n. For large X, the $1/L$ function levels off asymptotically to a maximum as the armature is removed to infinity.

As a dynamic state variable, current "I" bears a complex dynamic relationship to voltage and position. As a dependent algebraic variable, "I" is simply the product of flux linkage "$n\Phi$" and the "$1/L$" function. Let us examine these relationships more closely. First, we repeat Eq. 21a:

21a] $V_i = \dfrac{d(IL)}{dt}$ (repeated)

This equation is correct for variable inductances, for example, solenoids, where $L = L(X)$. Recalling the physics of fixed linear inductors, one might have believed the following:

~~21a]~~ $V_i = L\dfrac{dI}{dt}$ **not true** when L varies with time

Expanding the derivative in Eq. 21a gives:

23] $V_i = L\dfrac{dI}{dt} + I\dfrac{dL}{dt}$

The first term of 23 is the familiar linear inductor equation. The second term of 23 is called "back-EMF":

24] $\text{back-EMF} \equiv I\dfrac{dL}{dt}$ definition of the term

Two equations for magnetic force are given here without derivation, which will be deferred to the Part 2 paper following this one. It is noted that the most "obvious" derivation of Eqs. 25 and 26 incorrectly gives the negatives of the force results shown here:

25] $F = \frac{1}{2}\, I^2\, \dfrac{\partial L}{\partial X}$ force in the "current and position" frame of reference

26] $F = -\frac{1}{2}\, (n\Phi)^2 \dfrac{\partial(1/L)}{\partial X}$ force in the "flux and position" frame of reference

Eq. 26 was previewed in Eq. 7'. Imagine a "perfect" solenoid for which the position "X" is chosen to represent the magnetic gap, so that as X goes to zero, inductance goes to infinity and the "$1/L$" function goes smoothly to zero with a finite slope. Since $\dfrac{d(n\Phi)}{dt} = V_i$ (Eq. 21b) and the inductive voltage V_i remains finite with no singularity at $X = 0$, we conclude that for an energized solenoid, $n\Phi$ must remain finite with no singularity at the solenoid

impact point, X = 0. In force equation 26, the slope of reciprocal inductance, $\frac{\partial(1/L)}{\partial X}$, is finite and continuous at X=0, as is the square of flux linkage, nΦ, so there is no singularity in force. The rate at which magnetic force can be altered by a control voltage is a smoothly varying function, highest near X=0 and highest where the flux linkage is high (giving a steep slope to the $(n\Phi)^2$ parabola), and declining smoothly for increasing X and decreasing nΦ. Finally, since current is I = nΦ • (1/L) (Eq. 6') and since the (1/L) function has a finite slope at X=0, we conclude that at X = 0, current I = 0 and $\frac{dI}{dt}$ is finite. Since the resistive voltage term "I R" in Eq. 20 goes continuously to zero at the impact point, we further conclude that at X = 0, inductive voltage V_i equals the applied voltage V. Thus, the dynamic system is "well behaved" right up to the moment of impact, with the applied voltage approaching total control of the time derivative of flux linkage on approach to impact.

This description is in stark contrast to the description commonly given for solenoid behavior on approach to closure. There is no need to single out any particular paper for the "classic" description, since the conclusions are implied by basic physics equations. Examining the terms of Eq. 23 for the idealized "impact at X = 0" scenario just described, we have shown that the left-hand term, inductive voltage V_i , approaches the finite driving voltage V at impact. The term "$L\frac{dI}{dt}$" must go to infinity, since L goes to infinity and $\frac{dI}{dt}$ is finite for non-zero flux linkage and non-zero velocity, meaning for an energized solenoid making a realizable imperfect landing. Since the two right-hand terms of Eq. 23 must sum to a finite number, the back-EMF term "$I\frac{dL}{dt}$ " must go to infinity in the opposite direction from "$L\frac{dI}{dt}$".

In force equation 25, $\frac{\partial L}{\partial X}$ goes to infinity in proportion to $\frac{1}{X^2}$ for small X, while the current is forced to zero by a skyrocketing back-EMF against a similarly skyrocketing inductive impedance – the "irresistible force" (back-EMF) meeting the "immovable object" (inductive impedance). Adding to the confusion, the force equation becomes infinitely sensitive to current, even as current comes under the control of overwhelming forces related to the closing motion of the solenoid, beyond the influence of servo control.

In a real solenoid, the singularity point is not quite reached, lying at a slightly negative X while impact stops the armature at X=0. At impact, the current is small but non-zero, the inductance and back-EMF are large but finite, and the sensitivity of force to current is large but finite. It is commonly believed that unless great care is exercised, a solenoid will be driven into magnetic saturation as it closes, since only a small current is needed to saturate the solenoid. It is also commonly believed that the steep rise in inductance near impact implies that servo control over magnetic force is lost as the solenoid closes.

These beliefs are largely confused. It is true that when a solenoid approaches closure, the time left for course corrections vanishes rapidly, while voltages intended to control position must "filter" their way through tree layers of dynamic integration (from voltage to flux and force, from force and acceleration to velocity, and from velocity to position). Except for the shrinking time window before impact, however, the "authority" of the control system over magnetic force is actually at a maximum as the solenoid approaches closure. Current is

indeed difficult to control near solenoid closure, but back-EMF tends to force current to zero at just the right rate to maintain steady rates-of-change in both magnetic force and flux.

Attempting to exert control based on measured position and measured current is problematic. In order to control motion, the real objective is not current control, but force control. To solve Eq. 25 accurately for force, one must use measured position X and a calibration function describing the force coefficient, $\frac{\partial L}{\partial X}$, whose slope is large and rapidly changing near X=0. Current must also be resolved near zero, where small absolute measurement errors become large fractional errors in the "I^2" term of Eq. 25. Thus, the computation of magnetic force from current and position is highly error-sensitive.

Flux Linkage Servo Control

The difficulties of mathematical representation and servo control of solenoids are largely resolved in a new frame of reference. We will recapitulate the mathematical conclusions briefly, them move on to the somewhat different demands of servo control.

Recapitulation of the Math:
- $n\Phi$, flux linkage, becomes a state variable (Eq. 1')
- I, current, becomes a dependent variable (Eq. 6')
- V_i, the *total inductive voltage*, becomes the focus of attention
- $I\frac{dL}{dt}$ = "back-EMF" becomes an unneeded "fictitious force"
- $n\Phi$ is computed by integration of V_i (Eq. 22)
- 1/L is the non-singular calibration function of a solenoid's electromechanical behavior
- I R = resistive voltage
- $V = V_i + I R$ is the total voltage that must drive the system
- F, magnetic force, is computed from non-singular Eq. 26

Servo Control Context:
- 1/L = 1/L(X) is calibrated to characterize the solenoid
- I, current, is measured
- I R, resistive voltage, is computed
- V, applied voltage, is known or measured
- V_i , inductive voltage, is computed from V and I R (Eq. 20)
- $n\Phi$, flux linkage, is computed by integration of V_i (Eq. 22)
- initial X is known, so $\frac{I}{n\Phi}$ ratio is known, therefore $n\Phi$ is known for the start of integration
- X, position, is subsequently computed from I and $n\Phi$ (from Eq. 6', using 1/L(X) calibration)
- F, magnetic force, is computed for the present and extrapolated future based on Eq. 26
- $\dot{X}$, velocity, is computed from recent change in X plus known and extrapolated force F
- V, applied voltage, is controlled to cause V_i and $n\Phi$ to be controlled, thus controlling force

Several important issues arise for servo control. The description above postulates "sensorless" control, meaning that armature position X is not sensed, nor is there a separate sense coil to measure inductive voltage. Current is measured and applied voltage is either measured or else known because it is controlled in a known way (see U.S. patent 6,249,418, "System for control of an electromagnetic actuator" by Bergstrom), so any sensing that takes place is removed from the solenoid and into the controller electronics. The difficulty with this approach is initialization and subsequent drift of the flux linkage integral.

For initialization of flux linkage, a solenoid generally starts from a known full-open position X, for which the ratio $\dfrac{I}{n\Phi}$ is known. If the solenoid is of soft ferromagnetic material, includes no permanent magnets, and is initially at zero current, then the initial flux linkage is trivially zero.

If a permanent magnet is included, the magnet behaves like a source of ampere-turns, which is added to the product "nI" to give a corrected value. Having low differential permeability, permanent magnets add a significant fixed reluctance, $\mathfrak{R}$, to the magnetic circuit, which affects inductance and is taken into account in the calibration function 1/L(X). Thus, permanent magnets are readily incorporated into the present formulation. With permanent magnets in a solenoid (for example, a self-latching bistable solenoid that is unlatched by a releasing current pulse), flux linkage is initialized for a known position, typically either full-open or full-closed, based on the correct 1/L(X) calibration and the known magnetomotive force of the permanent magnet.

For rotary motors, with angle θ replacing position X, the function 1/L(θ) will have identifiable maxima and minima, corresponding to known values of θ and known maximum and minimum values. Thus, the ratio $\dfrac{I}{n\Phi}$ is known at identifiable maxima and minima in the rotation, and the flux linkage can therefore be initialized from the known maximum or minimum ratio.

After initialization, integration drift is a potential problem. The resistive voltage I•R is critical, and R is subject to thermal drift. Fortunately, there are several opportunities for a controller to recalibrate the resistance R. Whenever the armature is not moving and a steady current is flowing through the winding, the applied voltage is equal to the resistive voltage, and it is possible to solve for R. An opportunity arises when a solenoid is latched. Another opportunity can be created when a solenoid is open, if the armature rests against a mechanical stop and does not move when the flux is raised to a fixed level below the threshold to start the armature moving. The controller can therefore calibrate R by ramping current up to a "probe" level, then holding current steady and computing R from the resulting steady voltage and current.

Integration drift is minimized but not eliminated by calibration of R. For a solenoid making a quick transit from one mechanical stop to another, the period of uncorrected integration is brief, so cumulative integration drift is not a problem. For a rotary motor, the flux linkage integral can be corrected each time 1/L(θ) passes through a maximum or minimum, where the ratio $\dfrac{I}{n\Phi}$ is known. This leaves only the situation of continuous servo control of solenoid armature position between latching points. This control problem is addressed in U.S. patent 6,300,733 by Bergstrom, "System to determine solenoid position

and flux without drift." This system is applicable where Pulse Width Modulation (PWM) is used to control the average voltage applied to a solenoid winding. Since PWM is the most efficient way to drive a solenoid, and since microprocessors and DSP controllers intended for electromagnetic control almost universally include PWM outputs, it is hardly a restriction to call for PWM control. For a known supply voltage and a known PWM duty cycle, the fundamental component of voltage ripple is known. Solenoid current ripple is measured. Allowing for high frequency effects such as eddy currents, the impedance of a solenoid at the PWM frequency differs from but bears a known relationship to the low-frequency inductance and, consequently, to the calibration function 1/L(X). Unfortunately, current ripple measurements are prone to be noisy. Filtered low-frequency PWM and current ripple data can provide a good low frequency calibration of flux linkage, down to DC. More rapid variations in flux linkage are tracked by integration of inductive voltage. Bergstrom provides a method for combining drift-immune current ripple data at low frequency with the low-noise flux linkage integral data at high frequency. The ripple data are strongly filtered, minimizing noise, and the resulting information is used to correct the drift of the flux linkage integration. Specifically, the integral drift is countered by adjusting the resistance estimate, R. This solution cancels slow integration drift and simultaneously corrects the integration of more rapid changes in the flux linkage, minimizing dynamic flux integration errors due to an error in R.

The derivations of this paper will segue into Part 2, which for those interested, will derive the magnetic force equations (Eqs. 25 and 26), and will also provide a semi-empirical formulation that yields an excellent fit for the function 1/L (X) for most solenoids.